8-4 Direct, Joint, and Inverse Variation

Main Ideas

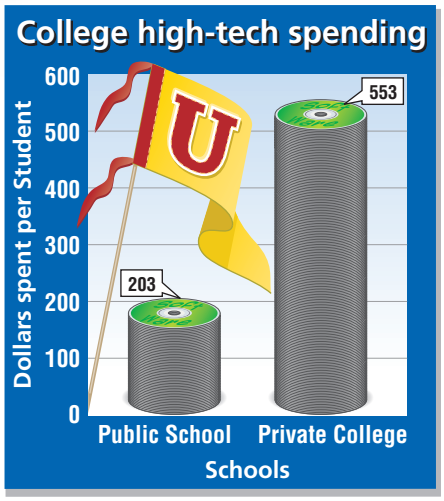
- Recognize and solve direct and joint variation problems.
- Recognize and solve inverse variation problems.

New Vocabulary

direct variation
constant of variation
joint variation
inverse variation

GET READY for the Lesson

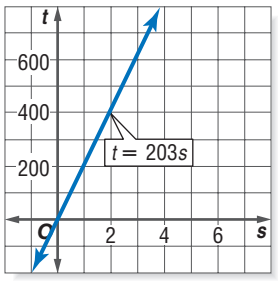
The total high-tech spending t of an average public college can be found by using the equation $t = 203s$, where s is the number of students.



Source: dthonline.com

Direct Variation and Joint Variation The relationship given by $t = 203s$ is an example of direct variation. A **direct variation** can be expressed in the form $y = kx$. The k in this equation is called the **constant of variation**.

Notice that the graph of $t = 203s$ is a straight line through the origin. An equation of a direct variation is a special case of an equation written in slope-intercept form, $y = mx + b$. When $m = k$ and $b = 0$, $y = mx + b$ becomes $y = kx$. So the slope of a direct variation equation is its constant of variation.



To express a direct variation, we say that y varies directly as x. In other words, as x increases, y increases or decreases at a constant rate.

KEY CONCEPT *Direct Variation*

y varies directly as x if there is some nonzero constant k such that $y = kx$. k is called the constant of variation.

If you know that y varies directly as x and one set of values, you can use a proportion to find the other set of corresponding values.

$$y_1 = kx_1 \quad \text{and} \quad y_2 = kx_2$$

$$\frac{y_1}{x_1} = k \qquad \frac{y_2}{x_2} = k \qquad \text{Therefore, } \frac{y_1}{x_1} = \frac{y_2}{x_2}.$$

Using the properties of equality, you can find many other proportions that relate these same x- and y-values.

EXAMPLE 1 Direct Variation

If y varies directly as x and $y = 12$ when $x = -3$, find y when $x = 16$.

Use a proportion that relates the values.

$\frac{y_1}{x_1} = \frac{y_2}{x_2}$ Direct proportion

$\frac{12}{-3} = \frac{y_2}{16}$ $y_1 = 12$, $x_1 = -3$, and $x_2 = 16$

$16(12) = -3(y_2)$ Cross multiply.

$192 = -3y_2$ Simplify.

$-64 = y_2$ Divide each side by -3.

When $x = 16$, the value of y is -64.

CHECK Your Progress

1. If r varies directly as s and $r = -20$ when $s = 4$, find r when $s = -6$.

Another type of variation is joint variation. **Joint variation** occurs when one quantity varies directly as the product of two or more other quantities.

KEY CONCEPT *Joint Variation*

y varies jointly as x and z if there is some nonzero constant k such that $y = kxz$.

If you know that y varies jointly as x and z and one set of values, you can use a proportion to find the other set of corresponding values.

$y_1 = kx_1z_1$ and $y_2 = kx_2z_2$

$\frac{y_1}{x_1z_1} = k$ $\frac{y_2}{x_2z_2} = k$ Therefore, $\frac{y_1}{x_1z_1} = \frac{y_2}{x_2z_2}$.

EXAMPLE 2 Joint Variation

Suppose y varies jointly as x and z. Find y when $x = 8$ and $z = 3$, if $y = 16$ when $z = 2$ and $x = 5$.

Use a proportion that relates the values.

$\frac{y_1}{x_1z_1} = \frac{y_2}{x_2z_2}$ Joint variation

$\frac{16}{5(2)} = \frac{y_2}{8(3)}$ $y_1 = 16$, $x_1 = 5$, $z_1 = 2$, $x_2 = 8$, and $z_2 = 3$

$8(3)(16) = 5(2)(y_2)$ Cross multiply.

$384 = 10y_2$ Simplify.

$38.4 = y_2$ Divide each side by 10.

When $x = 8$ and $z = 3$, the value of y is 38.4.

CHECK Your Progress

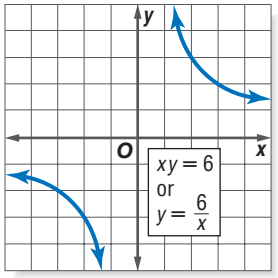
2. Suppose r varies jointly as s and t. Find r when $s = 2$ and $t = 8$, if $r = 70$ when $s = 10$ and $t = 4$.

Inverse Variation Another type of variation is inverse variation. For two quantities with **inverse variation**, as one quantity increases, the other quantity decreases. For example, speed and time for a fixed distance vary inversely with each other. When you travel to a particular location, as your speed increases, the time it takes to arrive at that location decreases.

KEY CONCEPT *Direct Variation*

y varies inversely as x if there is some nonzero constant k such that $xy = k$ or $y = \frac{k}{x}$, where $x \neq 0$ and $y \neq 0$.

Suppose y varies inversely as x such that $xy = 6$ or $y = \frac{6}{x}$. The graph of this equation is shown at the right. Since k is a positive value, as the values of x increase, the values of y decrease.

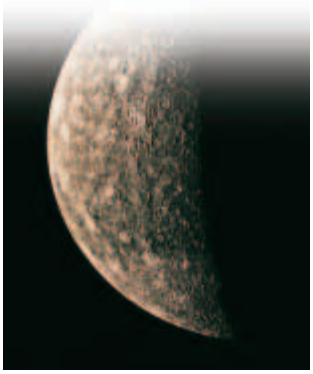


A proportion can be used with inverse variation to solve problems where some quantities are known. The following proportion is only one of several that can be formed.

$$x_1y_1 = k \text{ and } x_2y_2 = k$$

$$x_1y_1 = x_2y_2 \quad \text{Substitution Property of Equality}$$

$$\frac{x_1}{y_2} = \frac{x_2}{y_1} \quad \text{Divide each side by } y_1y_2.$$



Real-World Link

Mercury is about 36 million miles from the Sun, making it the closest planet to the Sun. Its proximity to the Sun causes its temperature to be as high as 800°F.

Source: *World Book Encyclopedia*

EXAMPLE Inverse Variation

3 **If r varies inversely as t and $r = 18$ when $t = -3$, find r when $t = -11$.**

$$\frac{r_1}{t_2} = \frac{r_2}{t_1} \quad \text{Use a proportion that relates the values.}$$

$$\frac{18}{-11} = \frac{r_2}{-3} \quad r_1 = 18,\ t_1 = -3, \text{ and } t_2 = -11$$

$$18(-3) = -11(r_2) \quad \text{Cross multiply.}$$

$$-54 = -11r_2 \quad \text{Simplify.}$$

$$4\frac{10}{11} = r_2 \quad \text{Divide each side by } -11.$$

CHECK Your Progress

3. If x varies inversely as y and $x = 24$ when $y = 4$, find x when $y = 12$.

Real-World EXAMPLE

4 **SPACE** **The apparent length of an object is inversely proportional to one's distance from the object. Earth is about 93 million miles from the Sun. Use the information at the left to find how many times as large the diameter of the Sun would appear on Mercury than on Earth.**

Explore The apparent diameter of the Sun varies inversely with the distance from the Sun. You know Mercury's distance from the Sun and Earth's distance from the Sun. You want to know how much larger the diameter of the Sun appears on Mercury than on Earth.

Plan Let the apparent diameter of the Sun from Earth equal 1 unit and the apparent diameter of the Sun from Mercury equal m. Then use a proportion that relates the values.

Solve

$$\frac{\text{distance from Mercury}}{\text{apparent diameter from Earth}} = \frac{\text{distance from Earth}}{\text{apparent diameter from Mercury}}$$ Inverse variation

$$\frac{36 \text{ million miles}}{1 \text{ unit}} = \frac{93 \text{ million miles}}{m \text{ units}}$$ Substitution

$$(36 \text{ million miles})(m \text{ units}) = (93 \text{ million miles})(1 \text{ unit})$$ Cross multiply.

$$m = \frac{(93 \text{ million miles})(1 \text{ unit})}{36 \text{ million miles}}$$ Divide each side by 36 million miles.

$$m \approx 2.58 \text{ units}$$ Simplify.

Check Since the distance between the Sun and Earth is between 2 and 3 times the distance between the Sun and Mercury, the answer seems reasonable. From Mercury, the diameter of the Sun will appear about 2.58 times as large as it appears from Earth.

CHECK Your Progress

4. SPACE Jupiter is about 483.6 million miles from the Sun. Use the information above to find how many times as large the diameter of the Sun would appear on Earth as on Jupiter.

Online **Personal Tutor at** algebra2.com

CHECK Your Understanding

Examples 1–3 (pp. 466–467)

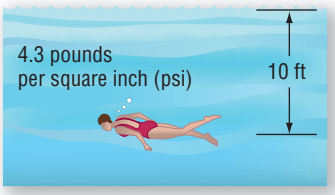
1. If y varies directly as x and $y = 18$ when $x = 15$, find y when $x = 20$.

2. Suppose y varies jointly as x and z. Find y when $x = 9$ and $z = -5$, if $y = -90$ when $z = 15$ and $x = -6$.

3. If y varies inversely as x and $y = -14$ when $x = 12$, find x when $y = 21$.

Example 4 (pp. 467–468)

SWIMMING For Exercises 4–7, use the following information.
When a person swims underwater, the pressure in his or her ears varies directly with the depth at which he or she is swimming.



4. Write a direct variation equation that represents this situation.

5. Find the pressure at 60 feet.

6. It is unsafe for amateur divers to swim where the water pressure is more than 65 pounds per square inch. How deep can an amateur diver safely swim?

7. Make a table showing the number of pounds of pressure at various depths of water. Use the data to draw a graph of pressure versus depth.

Exercises

HOMEWORK HELP	
For Exercises	See Examples
8, 9	1
10, 11	2
12, 13	3
14, 15	4

8. If y varies directly as x and $y = 15$ when $x = 3$, find y when $x = 12$.

9. If y varies directly as x and $y = 8$ when $x = 6$, find y when $x = 15$.

10. Suppose y varies jointly as x and z. Find y when $x = 2$ and $z = 27$, if $y = 192$ when $x = 8$ and $z = 6$.

11. If y varies jointly as x and z and $y = 80$ when $x = 5$ and $z = 8$, find y when $x = 16$ and $z = 2$.

12. If y varies inversely as x and $y = 5$ when $x = 10$, find y when $x = 2$.

13. If y varies inversely as x and $y = 16$ when $x = 5$, find y when $x = 20$.

14. GEOMETRY How does the circumference of a circle vary with respect to its radius? What is the constant of variation?

15. TRAVEL A map of Alaska is scaled so that 3 inches represents 93 miles. How far apart are Anchorage and Fairbanks if they are 11.6 inches apart on the map?

Geoff Butler

Real-World Career

Travel Agent

Travel agents give advice and make arrangements for transportation, accommodations, and recreation. For international travel, they also provide information on customs and currency exchange.

Math Online

For more information, go to algebra2.com.

State whether each equation represents a *direct*, *joint*, or *inverse* variation. Then name the constant of variation.

16. $\frac{n}{m} = 1.5$

17. $3 = \frac{a}{b}$

18. $a = 5bc$

19. $V = \frac{1}{3}Bh$

20. $p = \frac{12}{q}$

21. $\frac{2.5}{t} = s$

22. $vw = -18$

23. $y = -7x$

24. $V = \pi r^2 h$

25. If y varies directly as x and $y = 9$ when x is -15, find y when $x = 21$.

26. If y varies directly as x and $x = 6$ when $y = 0.5$, find y when $x = 10$.

27. Suppose y varies jointly as x and z. Find y when $x = \frac{1}{2}$ and $z = 6$, if $y = 45$ when $x = 6$ and $z = 10$.

28. If y varies jointly as x and z and $y = \frac{1}{8}$ when $x = \frac{1}{2}$ and $z = 3$, find y when $x = 6$ and $z = \frac{1}{3}$.

29. If y varies inversely as x and $y = 2$ when $x = 25$, find x when $y = 40$.

30. If y varies inversely as x and $y = 4$ when $x = 12$, find y when $x = 5$.

31. CHEMISTRY Boyle's Law states that when a sample of gas is kept at a constant temperature, the volume varies inversely with the pressure exerted on it. Write an equation for Boyle's Law that expresses the variation in volume V as a function of pressure P.

32. CHEMISTRY Charles' Law states that when a sample of gas is kept at a constant pressure, its volume V will increase directly as the temperature t. Write an equation for Charles' Law that expresses volume as a function.

LAUGHTER For Exercises 33–35, use the following information.

A newspaper reported that the average American laughs 15 times per day.

33. Write an equation to represent the average number of laughs produced by m household members during a period of d days.

34. Is your equation in Exercise 33 a *direct*, *joint*, or *inverse* variation?

35. Assume that members of your household laugh the same number of times each day as the average American. How many times would the members of your household laugh in a week?

Real-World Link

In order to sustain itself in its cold habitat, a Siberian tiger requires 20 pounds of meat per day.

Source: *Wildlife Fact File*

BIOLOGY **For Exercises 36–38, use the information at the left.**

36. Write an equation to represent the amount of meat needed to sustain s Siberian tigers for d days.

37. Is your equation in Exercise 36 a *direct*, *joint*, or *inverse* variation?

38. How much meat do three Siberian tigers need for the month of January?

39. **WORK** Paul drove from his house to work at an average speed of 40 miles per hour. The drive took him 15 minutes. If the drive home took him 20 minutes and he used the same route in reverse, what was his average speed going home?

40. **WATER SUPPLY** Many areas of Northern California depend on the snowpack of the Sierra Nevada Mountains for their water supply. If 250 cubic centimeters of snow will melt to 28 cubic centimeters of water, how much water does 900 cubic centimeters of snow produce?

41. **RESEARCH** According to Johannes Kepler's third law of planetary motion, the ratio of the square of a planet's period of revolution around the Sun to the cube of its mean distance from the Sun is constant for all planets. Verify that this is true for at least three planets.

ASTRONOMY **For Exercises 42–44, use the following information.**

Astronomers can use the brightness of two light sources, such as stars, to compare the distances from the light sources. The intensity, or brightness, of light I is inversely proportional to the square of the distance from the light source d.

42. Write an equation that represents this situation.

43. If d is the independent variable and I is the dependent variable, graph the equation from Exercise 42 when $k = 16$.

44. If two people are viewing the same light source, and one person is three times the distance from the light source as the other person, compare the light intensities that the two people observe.

GRAVITY **For Exercises 45–47, use the following information.**

According to the Law of Universal Gravitation, the attractive force F in Newtons between any two bodies in the universe is directly proportional to the product of the masses m_1 and m_2 in kilograms of the two bodies and inversely proportional to the square of the distance d in meters between the bodies. That is, $F = G\frac{m_1 m_2}{d_2}$. G is the universal gravitational constant. Its value is 6.67×10^{-11} Nm^2/kg^2.

45. The distance between Earth and the Moon is about 3.84×10^8 meters. The mass of the Moon is 7.36×10^{22} kilograms. The mass of Earth is 5.97×10^{24} kilograms. What is the gravitational force that the Moon and Earth exert upon each other?

46. The distance between Earth and the Sun is about 1.5×10^{11} meters. The mass of the Sun is about 1.99×10^{30} kilograms. What is the gravitational force that the Sun and Earth exert upon each other?

47. Find the gravitational force exerted on each other by two 1000-kilogram iron balls a distance of 0.1 meter apart.

EXTRA PRACTICE
See pages 908, 933.
Math Online
Self-Check Quiz at algebra2.com

H.O.T. Problems

48. OPEN ENDED Describe two real life quantities that vary directly with each other and two quantities that vary inversely with each other.

49. CHALLENGE Write a real-world problem that involves a joint variation. Solve the problem.

50. *Writing in Math* Use the information about variation on page 465 to explain how variation is used to determine the total cost if you know the unit cost.

STANDARDIZED TEST PRACTICE

51. ACT/SAT Suppose b varies inversely as the square of a. If a is multiplied by 9, which of the following is true for the value of b?

A It is multiplied by $\frac{1}{3}$.

B It is multiplied by $\frac{1}{9}$.

C It is multiplied by $\frac{1}{81}$.

D It is multiplied by 3.

52. REVIEW If $ab = 1$ and a is less than 0, which of the following statements cannot be true?

F b is negative.

G b is less than a.

H As a increases, b decreases.

J As a increases, b increases.

Spiral Review

Determine the equations of any vertical asymptotes and the values of x for any holes in the graph of each rational function. (Lesson 8-3)

53. $f(x) = \dfrac{x+1}{x^2-1}$ **54.** $f(x) = \dfrac{x+3}{x^2+x-12}$ **55.** $f(x) = \dfrac{x^2+4x+3}{x+3}$

Simplify each expression. (Lesson 8-2)

56. $\dfrac{3x}{x-y} + \dfrac{4x}{y-x}$ **57.** $\dfrac{t}{t+2} - \dfrac{2}{t^2-4}$ **58.** $\dfrac{m-\frac{1}{m}}{1+\frac{4}{m}-\frac{5}{m^2}}$

59. BIOLOGY One estimate for the number of cells in the human body is 100,000,000,000,000. Write this number in scientific notation. (Lesson 6-1)

State the slope and the y-intercept of the graph of each equation. (Lesson 2-4)

60. $y = 0.4x + 1.2$ **61.** $2y = 6x + 14$ **62.** $3x + 5y = 15$

GET READY for the Next Lesson

PREREQUISITE SKILL Identify each function as S for step, C for constant, A for absolute value, or P for piecewise. (Lesson 2-6)

63. $h(x) = \dfrac{2}{3}$ **64.** $g(x) = 3|x|$ **65.** $f(x) = [\![2x]\!]$

66. $f(x) = \begin{cases} 1 \text{ if } x > 0 \\ -1 \text{ if } x \le 0 \end{cases}$ **67.** $h(x) = |x-2|$ **68.** $g(x) = -3$